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Phase Transitions on Fractal Lattices with Long-Range Interactions

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A fractal lattice F is defined here to comprise all points of the form $\mathbf{a} + m\mathbf{a}' + m^2\mathbf{a}'' + \cdots + m^q\mathbf{a}^{(q)}$, where q is a nonnegative integer and $\mathbf{a}, \mathbf{a}', ..., \mathbf{a}^{(q)} \in A$, where A is a finite set of points in some Euclidean space. Provided m is not too small (in particular, m must be at least 2), the dimension of F is shown to be $D = \log n/\log m$, where n is the number of points in A. It is shown further that an Ising model on F, with a ferromagnetic pair interaction $r^{-\alpha}$ between spins separated by a distance r, has a phase transition if $D < \alpha < 2D$. On the other hand, for $\alpha > 2D$, provided a certain condition which rules out periodic lattices is satisfied, there can be no finite-temperature transition leading to spontaneous magnetization.

KEY WORDS: Phase transitions; fractal lattices; long-range forces; Ising ferromagnet.

1. INTRODUCTION

The effect of dimensionality on phase transitions has long been a topic of interest in statistical mechanics; indeed, according to the universality hypothesis, many of the salient features of the phase transition in a lattice system with short-range forces are entirely determined by two parameters: the number of dimensions of the local spin variables, and the number of dimensions of the lattice itself. The study of this dependence on lattice dimensionality was given an added twist by the discovery of Wilson and Fisher[(1)] that in some types of calculation the dimensionality of the lattice could be formally treated as a continuous variable capable of nonintegral

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values, such as 3.99. However (in the physics community, at least) it did not become clear until the work of Mandelbrot[2,3] that it was actually possible to define objects (which he called *fractals*) having nonintegral dimensionality.

Recently, Mandelbrot's ideas have stimulated some investigations of phase transitions on what may be called *fractal lattices.* These are infinite sets of points (called here *sites*) in some underlying Euclidean space, arranged in such a way that the number of sites within a distance R of any given site increases asymptotically in proportion to R^D, where D is a constant called the *fractal dimension* (or Hausdorff dimension). The fractal dimension need not be an integer; examples where it is not, and a general method for constructing such lattices, will be discussed in Section 2. Most of the studies carried out so far[4–8] have been for Ising models with interactions only between nearest neighbor sites (that is, sites that are neighbors in the underlying lattice). The results indicate a breakdown of simple universality: two Ising models with the same value of D can have different critical exponents, or one of them may have a positive-temperature phase transition, while the other has none. It may be possible to restore universality by using some other dimensionlike number—order of ramification, or connectivity dimension[4,8,9] perhaps—in place of the fractal dimension D, but the way to do this is not clear at present.

It is not really surprising that the fractal dimension D tells us too little about the properties of a nearest neighbor fractal Ising model to determine its phase transition characteristics. The definition of fractal dimension depends on the distribution of pairs of widely separated sites; it says nothing about the distribution of pairs of sites that are close together, in particular pairs of sites that are nearest neighbors on the underlying lattice. It is not even sensitive enough to distinguish an Ising model on (say) a cubic lattice from one that uses only alternate sites of such a lattice—yet if the interactions are confined to pairs of sites that are nearest neighbors on the underlying lattice, the former has a phase transition, while the latter has none.

As a possible indicator of phase transitions, the fractal dimension therefore has a better chance of success if we use it not for a nearest neighbor model, but for one with long-range interactions. Accordingly, the present paper is concerned with an interaction falling off as an inverse power of the distance. The system we consider is an Ising ferromagnet in which any finite set Λ of occupied lattice sites has the Hamiltonian

$$H_\Lambda = -\sum_{\{\mathbf{x},\mathbf{y}\} \subset \Lambda} s(\mathbf{x})\, s(\mathbf{y})\, |\mathbf{x}-\mathbf{y}|^{-\alpha} \tag{1.1}$$

where the sum goes over all unordered pairs $\{\mathbf{x}, \mathbf{y}\}$ of different sites in Λ,

$s(\mathbf{x})$ (capable of the values ± 1 only) is the spin at the site $\mathbf{x}$, $|\mathbf{x}-\mathbf{y}|$ is the distance between site $\mathbf{x}$ and site $\mathbf{y}$, and α is a positive constant. Similar results also apply to the plane rotator ferromagnet, as we shall see at the end of the paper.

An indication of how the value of α may be expected to affect the phase transitions of our model can be obtained by considering some perturbations of the totally ordered state defined to have $s(\mathbf{x}) = +1$ for all sites $\mathbf{x}$ in the infinite fractal lattice. (The following argument is a natural generalization of one given for the two-dimensional plane rotator model by Kunz and Pfister.[10]) First, consider the energy required to reverse just one spin in this state; this energy may be estimated, using the definition of D, as being roughly

$$\text{const} \times \int_1^\infty r^{-\alpha}\, d(r^D)$$

In the case $\alpha \leqslant D$ this integral diverges; it takes an infinite energy to reverse even one spin and so the system will be totally ordered at all temperatures. We therefore expact to find a phase transition only in the other case, $\alpha > D$. Second (supposing now that $\alpha > D$), consider the energy required to reverse all the spins in a sphere S of radius R (centered on an arbitrary site); this energy falls short of the sum of the energies required to reverse each one of them separately by an amount

$$4 \sum_{\{\mathbf{x},\mathbf{y}\} \subset S} |\mathbf{x}-\mathbf{y}|^{-\alpha} \tag{1.2}$$

(In the equivalent lattice gas description of the Ising model this quantity is the binding energy of a spherical droplet of radius R.) Since $|\mathbf{x}-\mathbf{y}| \leqslant 2R$, this energy difference is bounded below by $4(2R)^{-\alpha}$ times the number of pairs of occupied sites in S, that is, by a quantity proportional to $R^{2D-\alpha}$ for large R. So if $\alpha < 2D$ there is a lot of energy to be gained by collecting spins together in large clusters rather than scattering the same number of spins at random over the whole system; hence the spins will tend to segregate into large clusters, and we expect to find phase separation at sufficiently low temperatures. These considerations lead us to expect that the infinite system will have a phase transition if α lies between D and $2D$. A proof of this is the main result of the present paper.

The method of proof depends strongly on the work of Dyson,[11] who proved that, if instead of a fractal lattice we use the infinite one-dimensional lattice Z, then the infinite system with Hamiltonian (1.1) has a phase transition for $1 < \alpha < 2$. The main idea of Dyson's proof was to use the inequalities of Griffiths[12,13] to compare this system with another system, which he called the *hierarchical model*, for which the existence of a low-temperature spontaneous magnetization can be proved if $\alpha < 2$. At the same time a different inequality, also due to Griffiths,[14] enabled Dyson to show that there is no spontaneous magnetization at high temperatures if $\alpha > 1$. So if $1 < \alpha < 2$, there is spontaneous magnetization at low temperatures, but not at high: a phase transition. The present paper uses the same ideas: we shall find that a similar comparison can be made in the case of a fractal lattice with arbitrary dimension D, and that there is, as the argument given above suggests, a phase transition provided

$$D < \alpha < 2D \tag{1.3}$$

In this way Dyson's result will be generalized to cases where $D \neq 1$.

2. FRACTAL LATTICES

The fractal lattices to be considered in this paper can all be constructed in the following way: start with a finite *generating set* A comprising n points (with $n \geqslant 2$) in some Euclidean space; expand it by an arbitrary factor m (with $m > 1$); replace every point of the expanded set by a replica of A to give a new set A'; expand again by a factor m, replace every point of the new expanded set by a replica of A to give a new set A''; and so on. Figure 1 illustrates the first four steps in this procedure for the case where A consists of the points 0 and 1 on the real line, with $m = 3$. Another example, the Sierpinski carpet, is illustrated in Fig. 2.

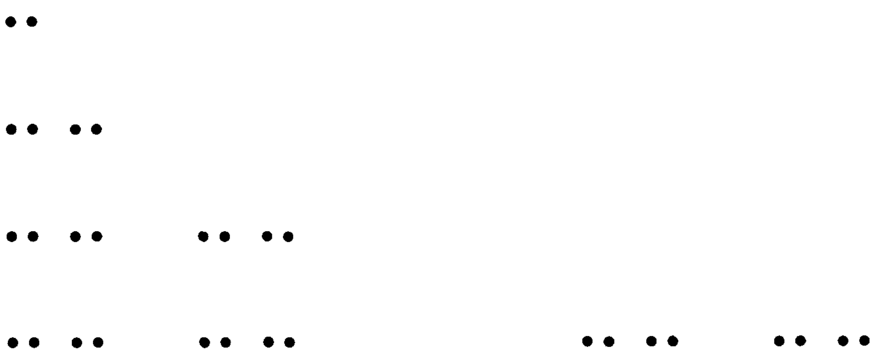
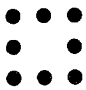
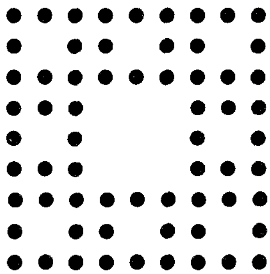


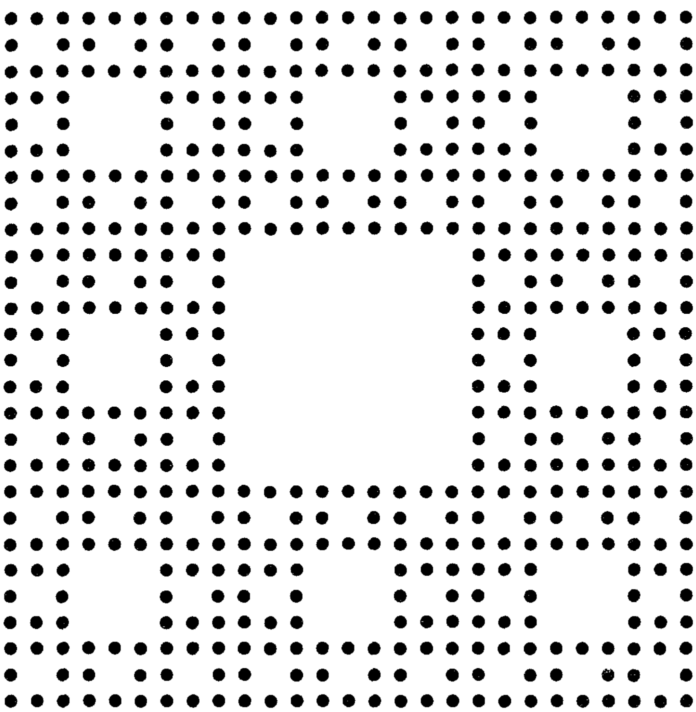
Fig. 1. The first four steps in the construction of a fractal lattice. The first line shows the generating set A; the second line is A', the third is A'', and the fourth is $A^{(3)}$. For this lattice we have $n = 2$, $m = 3$.



(a)



(b)



(c)

Fig. 2. The sets A, A', and A'' for a "Sierpinski carpet" fractal lattice. For this lattice we have $n = 8$, $m = 3$.

To obtain a formula for the resulting fractal lattice (call it F), we represent the points comprised in A by vectors referred to one of these points as origin:

$$A = \{\mathbf{0}, \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n-1}\} \qquad (n \geqslant 2) \tag{2.1}$$

where n is the number of points in A. Then A' consists of vectors of the form $\mathbf{a} + m\mathbf{a}'$ with $\mathbf{a}, \mathbf{a}' \in A$; A'' consists of vectors of the form $\mathbf{a} + m\mathbf{a}' + m^2\mathbf{a}''$; etc; and so we have, for any nonnegative integer q,

$$A^{(q)} = \{\mathbf{a} + m\mathbf{a}' + \cdots + m^q\mathbf{a}^{(q)}\colon\ \mathbf{a}, \mathbf{a}', \ldots, \mathbf{a}^{(q)} \in A\} \tag{2.2}$$

The fractal lattice F can then be defined as

$$F = \bigcup_{q=0}^{\infty} A^{(q)} \tag{2.3}$$

An equivalent formulation is that F consists of all convergent series of the form $\sum_{p \geqslant 0} m^p \mathbf{a}^{(p)}$ with $\mathbf{a}^{(p)} \in A$ for all p, since with $m > 1$ the series converges if and only if $\mathbf{a}^{(p)} = \mathbf{0}$ for all sufficiently large p. Incidentally, this latter way of defining F can also be used to define fractal sets, instead of fractal lattices, by taking $m < 1$ instead of $m > 1$; for example, if we take $A = \{0, 2/3\}$ and $m = 1/3$, then F is the Cantor set.

Not every object to which the term "fractal lattice" might be applied falls precisely within the above definition. An example is the lattice corresponding to the Sierpinski gasket. This lattice can be constructed by taking the points of A to be the vertices of an equilateral triangle, but using in place of (2.2) the formula

$$A^{(q)} = \{\mathbf{a} + \mathbf{a}' + 2\mathbf{a}'' + \cdots + 2^{q-1}\mathbf{a}^{(q)}\colon \mathbf{a}, \ldots, \mathbf{a}^{(q)} \in A\}$$

In the present paper, however, only fractal lattices of the particular form (2.2) will be considered.

To avoid unnecessary complications in the theory that follows, it is desirable to ensure that the points constructed by the above method are all distinct, and that none of them are very close together even when q is very large. This will be achieved here by requiring not just $m > 1$, but

$$m \geqslant 1 + r_{\max}/r_{\min} \tag{2.4}$$

where $r_{\max}$ is the diameter of the set A and $r_{\min}$ is the least distance between pairs of points in A, measured using some suitable norm. This norm need not be the Euclidean one; for example, if we build up a plane square lattice by taking $m=2$ and A the four points at the corners of a square, then (2.4) is not satisfied for the Euclidean norm, but is satisfied for the norm

$$\|x\mathbf{i}+y\mathbf{j}\| = \max(|x|, |y|)$$

with $\mathbf{i}$ and $\mathbf{j}$ unit vectors along the sides of the square. Incidentally, the value of m used in this example is the least possible, since (2.4) implies $m \geqslant 2$.

Theorem 1. If (2.4) holds, then (a) every point $\mathbf{x}$ of F has a unique representation in the form

$$\mathbf{x} = \sum_0^q m^p \mathbf{x}^{(p)} \qquad (\mathbf{x}^{(0)}, \mathbf{x}^{(1)},..., \mathbf{x}^{(q)} \in A; \mathbf{x}^{(q)} \neq \mathbf{0}) \tag{2.5}$$

and (b) the distance between any pair of points in F is at least $r_{\min}$.

Proof. Let $\mathbf{x}$ be any point of F; then by (2.2) it can be represented in the form (2.5). Let $\mathbf{y}$ be a point whose representation is different, say

$$\mathbf{y} = \sum_0^r m^p \mathbf{y}^{(p)} \qquad (\mathbf{y}^{(0)}, \mathbf{y}^{(1)},..., \mathbf{y}^{(r)} \in A; \mathbf{y}^{(r)} \neq \mathbf{0})$$

Then we have, for some nonnegative integer s not exceeding the greater of q and r,

$$\mathbf{x} - \mathbf{y} = \sum_0^s m^p (\mathbf{x}^{(p)} - \mathbf{y}^{(p)}) \qquad (\mathbf{x}^{(s)} \neq \mathbf{y}^{(s)})$$

The norm of $\mathbf{x} - \mathbf{y}$ therefore satisfies

$$\begin{aligned}
\|\mathbf{x} - \mathbf{y}\| &\geqslant m^s r_{\min} - (m^{s-1} + m^{s-2} + \cdots + m + 1) r_{\max} \\
&= m^s r_{\min} - \frac{m^s - 1}{m - 1} r_{\max} \\
&\geqslant m^s r_{\min} - (m^s - 1) r_{\min} \qquad [\text{by } (2.4)] \\
&= r_{\min}
\end{aligned}$$

Hence (a) two points of F with different representations cannot coincide and (b) two distinct points of F are separated by at least $r_{\min}$. ∎

3. THE DIMENSION OF A FRACTAL LATTICE

To define the dimension of a fractal lattice F let $R_N(\mathbf{x})$ denote, for each positive integer N and each $\mathbf{x}$ in F, the radius of the smallest closed ball, centered at the point $\mathbf{x}$, that includes at least N points of F. Then, if $R_N(\mathbf{x}) \sim \text{const} \times N^{1/D}$ for large N [in the sense that $N^{-1/D} R_N(\mathbf{x})$ has positive upper and lower bounds], we shall say that D is the dimension of F. The following theorem gives the value of D and shows that these upper and lower bounds can be chosen independent of $\mathbf{x}$.

Theorem 2. For any fractal lattice F there exist positive constants K_1 and K_2 such that

$$K_1 \leqslant N^{-1/D} R_N(\mathbf{x}) \leqslant K_2 \qquad \text{for all } \mathbf{x} \in F \text{ and for all } N \geqslant n$$

where

$$D = \log n / \log m \tag{3.1}$$

n being the number of points in the generating set and m the magnification factor.

Proof.[1] For each positive integer q, define the positive quantity $\rho_q(\mathbf{x})$ by

$$\rho_q(\mathbf{x})^2 = \frac{1}{n^{q+1}} \sum_{\mathbf{y}} (\mathbf{y} - \mathbf{x})^2 \tag{3.2}$$

where the sum goes over all the n^{q+1} points $\mathbf{y}$ in $\mathbf{A}^{(q)}$ and $(\mathbf{y} - \mathbf{x})^2$ means the Euclidean scalar product of $\mathbf{y} - \mathbf{x}$ with itself.

Using the representation $\mathbf{y} = \mathbf{a} + m\mathbf{a}' + \cdots + m^q \mathbf{a}^{(q)}$, we calculate

$$\begin{aligned}\rho_q(\mathbf{x})^2 &= \frac{1}{n^{q+1}} \sum_{\mathbf{a} \in A} \sum_{\mathbf{a}' \in A} \cdots \sum_{\mathbf{a}^{(q)} \in A} (\mathbf{a} + m\mathbf{a}' + \cdots + m^q \mathbf{a}^{(q)} - \mathbf{x})^2 \\ &= [(1 + m + \cdots + m^q)\boldsymbol{\mu}_1 - \mathbf{x}]^2 + (1 + m^2 + m^4 + \cdots + m^{2q})\,\mu_2\end{aligned}$$

where

$$\boldsymbol{\mu}_1 = \frac{1}{n} \sum_{\mathbf{a} \in A} \mathbf{a}$$

$$\mu_2 = \frac{1}{n} \sum_{\mathbf{a} \in A} (\mathbf{a} - \boldsymbol{\mu}_1)^2$$

$$> 0 \quad \text{since } n \geqslant 2$$

Noting, with the help of (2.5), that the modulus of the vector in square brackets is at most $(1 + m + \cdots + m^q) r_{\max}$, and using also the fact that $m \geqslant 2$, we obtain

$$m^{2q} \mu_2 \leqslant \rho_q(\mathbf{x})^2 \leqslant (m^{q+1} r_{\max})^2 + m^{2q+1} \mu_2$$

[1] Note added in proof: The proof given here for Theorem 2 is wrong. A corrected version of the theorem will be published separately.

It is therefore possible to find positive constants λ_1 and λ_2 such that

$$\lambda_1 m^q < \rho_q(\mathbf{x}) < \lambda_2 m^q \tag{3.3}$$

for all $\mathbf{x} \in F$ and all positive integers q.

Now we relate this behavior of $\rho_q(\mathbf{x})$ to the behavior of $R_N(\mathbf{x})$. For every N we have from (3.2) the Chebyshev-type inequality

$$\begin{aligned}\rho_q(\mathbf{x})^2 &\geqslant \frac{1}{n^{q+1}} \sum_{\mathbf{y} \in A^{(q)}:(\mathbf{y}-\mathbf{x})^2 > R_N(\mathbf{x})^2} (\mathbf{y}-\mathbf{x})^2 \\ &\geqslant \frac{R_N(\mathbf{x})^2}{n^{q+1}} [n^{q+1} - N]\end{aligned}$$

since, by the definition of $R_N(\mathbf{x})$, the truncated sum has at most $n^{q+1} - N$ terms.

Giving N any value not exceeding n^q, we find (since $n > 1$)

$$R_N(\mathbf{x}) \leqslant \frac{\rho_q(\mathbf{x})}{(1 - 1/n)^{1/2}} \qquad \text{if} \quad N \leqslant n^q \tag{3.4}$$

At the same time, since the sum in (3.2) contains n^{q+1} terms, the definition of $R_N(\mathbf{x})$ implies that no term in that sum can exceed $R_N(\mathbf{x})^2$ for any $N \geqslant n^{q+1}$. Consequently, (3.2) implies

$$\rho_q(\mathbf{x})^2 \leqslant R_N(\mathbf{x})^2 \qquad \text{if} \quad N \geqslant n^{q+1}$$

Replacing q by $q-2$ and rearranging, we find

$$R_N(\mathbf{x}) \geqslant \rho_{q-2}(\mathbf{x}) \qquad \text{if} \quad N \geqslant n^{q-1} \quad \text{and} \quad q \geqslant 2 \tag{3.5}$$

For every $N \geqslant n$ we may choose $q \geqslant 2$ so that

$$n^{q-1} \leqslant N \leqslant n^q$$

which is equivalent to

$$N^{1/D} \leqslant m^q \leqslant m N^{1/D}$$

with D given by (3.1). Combining this last result with (3.3), and then using (3.4) and (3.5), we find that

$$\lambda_1 N^{1/D} m^{-2} \leqslant R_N(\mathbf{x}) \leqslant \frac{\lambda_2 m N^{1/D}}{(1 - 1/n)^{1/2}} \qquad \text{if} \quad N \geqslant n$$

This completes the proof of Theorem 2. ▮

4. THERMODYNAMIC LIMIT

To define a thermodynamic limit for an Ising ferromagnet on the fractal lattice F, consider the sequence $A, A',..., A^{(q)},...$ of finite subsets of F defined in Section 2. With each of these subsets $A^{(q)}$ we can associate an Ising model in which only the spins within $A^{(q)}$ interact; that is, its Hamiltonian, in the notation of (1.1), is $H_{A^{(q)}}$. The correlation functions for the infinite lattice can then be defined as the limit of the correlation functions for the systems in this sequence; in particular, for the two-body function, we define

$$\langle s(\mathbf{x})\, s(\mathbf{y})\rangle = \lim_{q\to\infty} \langle s(\mathbf{x})\, s(\mathbf{y})\rangle^{(q)} \tag{4.1}$$

where $\langle\cdots\rangle^{(q)}$ means an average in the Ising model on $A^{(q)}$, with Hamiltonian

$$-\sum_{\{\mathbf{x},\mathbf{y}\}\subset A^{(q)}} s(\mathbf{x})\, s(\mathbf{y})\, J(\mathbf{x},\mathbf{y}) \tag{4.2}$$

where $J(\mathbf{x},\mathbf{y}) = |\mathbf{x}-\mathbf{y}|^{-\alpha}$. The existence of the limit in (4.1) is guaranteed by the argument of Griffiths,[15] i.e., the fact that in proceeding along the sequence we never weaken any interaction, so that by the Griffiths inequality[12] the bounded sequence $\langle s(\mathbf{x})\, s(\mathbf{y})\rangle^{(q)}$ is monotonic nondecreasing in q at fixed $\mathbf{x}, \mathbf{y}$. It should be noted, however, that the correlation functions need not be translationally invariant.

5. SPONTANEOUS MAGNETIZATION ABSENT AT HIGH TEMPERATURES

Following Dyson,[11] we can divide the proof that the system has a phase transition into two parts: a proof that it has no spontaneous magnetization at high temperatures if $\alpha > D$, and a proof that it does have spontaneous magnetization at low temperatures if $\alpha < 2D$. The first part of the proof follows closely the method of Dyson,[11] which starts from the inequality of Griffiths[14] for an Ising model in thermal equilibrium in zero magnetic field:

$$\langle s(\mathbf{x})\, s(\mathbf{y})\rangle^{(q)} \leqslant \tanh \beta J(\mathbf{x},\mathbf{y}) + \sum_{\substack{\mathbf{z}\in A^{(q)}\\ \mathbf{z}\neq \mathbf{x},\mathbf{y}}} \langle s(\mathbf{x})\, s(\mathbf{z})\rangle^{(q)} \tanh \beta J(\mathbf{z},\mathbf{y}) \tag{5.1}$$

where $\beta = 1/\kappa T$, with κ Boltzmann's constant and T the temperature, and

$J(\mathbf{x}, \mathbf{y}) = |\mathbf{x}-\mathbf{y}|^{-\alpha}$ is the interaction between the spins $s(\mathbf{x})$ and $s(\mathbf{y})$ on the sites $\mathbf{x}$ and $\mathbf{y}$. Since

$$0 \leqslant \langle s(\mathbf{x})\, s(\mathbf{z})\rangle^{(q)} \leqslant \langle s(\mathbf{x})\, s(\mathbf{z})\rangle$$

and $J(\mathbf{z}, \mathbf{y}) \geqslant 0$, Eq. (5.1) implies

$$\langle s(\mathbf{x})\, s(\mathbf{y})\rangle^{(q)}$$

$$\leqslant \tanh \beta J(\mathbf{x}, \mathbf{y}) + \sum_{\substack{z \in F \\ \mathbf{z} \neq \mathbf{x},\mathbf{y}}} \langle s(\mathbf{x})\, s(\mathbf{z})\rangle \tanh \beta J(\mathbf{z}, \mathbf{y}) \qquad (5.2)$$

provided the series converges. Since $\tanh \beta J \leqslant \beta J$, a sufficient condition for convergence of this series is convergence of the series $\sum_{\mathbf{z} \in F, \mathbf{z} \neq \mathbf{y}} J(\mathbf{z}, \mathbf{y})$, i.e., $\sum_{\mathbf{z} \in F, \mathbf{z} \neq \mathbf{y}} |\mathbf{z}-\mathbf{y}|^{-\alpha}$. Using the notation of Section 3, we can write this last series as $\int_0^\infty R_N(\mathbf{y})^{-\alpha}\, dN$; therefore, by the lower bound on $R_N(\mathbf{y})$ given in Theorem 2, this series converges if $\alpha > D$ and has an upper bound independent of $\mathbf{y}$; call the least such upper bound U. Taking the limit $q \to \infty$ in (5.2) and then summing over all $\mathbf{x}$ in an arbitrary finite subset S of F, we find

$$\sum_{x \in S} \langle s(\mathbf{x})\, s(\mathbf{y})\rangle \leqslant \beta U + \sum_{\mathbf{x} \in S} \sum_{\mathbf{z} \in F, \mathbf{z} \neq \mathbf{y}} \langle s(\mathbf{x})\, s(\mathbf{z})\rangle \tanh \beta J(\mathbf{z}, \mathbf{y})$$

$$\leqslant \beta U + \operatorname{Sup}_{\mathbf{z} \in F} \sum_{\mathbf{x} \in S} \langle s(\mathbf{x})\, s(\mathbf{z})\rangle\, \beta U$$

Hence, at temperatures high enough to make $\beta U < 1$, we have

$$\operatorname{Sup}_{\mathbf{y} \in F} \sum_{\mathbf{x} \in S} \langle s(\mathbf{x})\, s(\mathbf{y})\rangle \leqslant \frac{\beta U}{1-\beta U}$$

It follows, since $\langle s(\mathbf{x})\, s(\mathbf{y})\rangle \geqslant 0$ by Griffiths' inequality,[12] that the infinite sum $\sum_{\mathbf{x} \in F} \langle s(\mathbf{x})\, s(\mathbf{y})\rangle$ converges for every $\mathbf{y}$ and hence that $\langle s(\mathbf{x})\, s(\mathbf{y})\rangle$ approaches zero in the limit of large $|\mathbf{x}-\mathbf{y}|$. Thus we have proved the following result.

Theorem 3. If $\alpha > D$, the Ising ferromagnet with $r^{-\alpha}$ interactions on a fractal lattice F of dimension D has no spontaneous magnetization at temperatures T such that

$$\kappa T > U = \operatorname{Sup}_{\mathbf{y} \in F} \sum_{\substack{\mathbf{z} \in F \\ \mathbf{z} \neq \mathbf{y}}} |\mathbf{z}-\mathbf{y}|^{-\alpha}$$

6. DYSON'S HIERACHICAL MODEL

Our proof of spontaneous magnetization at low temperatures is based on a comparison with Dyson's hierarchical model.[11] This model is conveniently defined in terms of a function of two integers $d(i, j)$, defined as the position of the most significant place in which the binary representations of i and j differ. That is, if

$$i = \sum_{p \geqslant 0} 2^p i^{(p)} \quad \text{and} \quad j = \sum_{p \geqslant 0} 2^p j^{(p)} \tag{6.1}$$

with $i^{(p)}, j^{(p)} \in \{0, 1\}$, $(p = 0, 1, 2,...)$, then

$$\begin{aligned} &i^{(p)} = j^{(p)} \qquad \text{for all} \quad p \geqslant d(i, j) \\ \text{but} \\ &i^{(p)} \neq j^{(p)} \qquad \text{for} \quad p = d(i, j) - 1 \end{aligned} \tag{6.2}$$

[In Dyson's notation[11] our $d(i, j)$ would be called $p(i+1, j+1)$.]

A hierarchical Ising model is one in which the interaction between the ith and the jth spins is a function of $d(i, j)$ only. Dyson considered a sequence of finite hierarchical models $M_1, M_2,...,$ where the model M_N comprises 2^N spins $s_0, s_1,..., s_{2^N - 1}$ in which the interaction between the ith and jth spins is

$$\sum_{p = d(i,j)}^{N} 2^{1-2p} b_p \tag{6.3}$$

where $b_1, b_2,...$ are given positive constants. He showed that if

$$b_p = 2^{(2-\gamma)p} \qquad \text{with} \quad 1 < \gamma < 2 \tag{6.4}$$

then there is a positive temperature T_γ below which the system has a spontaneous magnetization, in the sense that

$$\lim_{|i-j| \to \infty} \langle s_i s_j \rangle > 0 \qquad \text{if} \quad T < T_\gamma \tag{6.5}$$

where $\langle s_i s_j \rangle$ denotes the limit, as $N \to \infty$, of the thermal average of $s_i s_j$ at temperature T in the finite hierarchical model M_N.

His proof also shows [Eq. (3.20) Ref. 11] that, for all i and j,

$$\langle s_i s_j \rangle \geqslant m^2(T) \tag{6.6}$$

where $m^2(T)$ is defined to be equal the left-hand side of (6.5). If follows, by Griffiths' inequality,[12] that the lower bound (6.6) on $\langle s_i s_j \rangle$ holds for any

Ising model in which the interactions are at least as strong as the ones given in (6.3) and (6.4)—in particular, one in which the interactions are, regardless of the value of N,

$$J_{ij} = \sum_{p=d(i,j)}^{\infty} 2^{1-2p} 2^{(2-\gamma)p} = c_\gamma 2^{-\gamma d(i,j)} \tag{6.7}$$

where

$$c_\gamma = 2/(1 - 2^{-\gamma}) \tag{6.8}$$

On rescaling the interactions and the temperature by a factor J_0/c_γ, where J_0 is an arbitrary positive number, this result takes the following form:

Lemma 4. If the sites in an Ising ferromagnet can be numbered in such a way that the interaction J_{ij} between the ith and jth sites satisfies

$$J_{ij} \geqslant J_0 2^{-\gamma d(i,j)}$$

with $1 < \gamma < 2$, then the correlations satisfy

$$\begin{aligned} \langle s_i s_j \rangle &\geqslant m^2(c_\gamma T/J_0) \\ &> 0 \qquad \text{if} \quad T < T_\gamma J_0/c_\gamma \end{aligned}$$

where $d(i, j)$ is defined in (6.1) and (6.2), m^2 just after (6.6), c_γ in (6.8), and T_γ in (6.5); note that $T_\gamma J_0/c_\gamma$ is positive.

This lemma will allow us to derive a lower bound on the correlation between two arbitrary sites in a fractal lattice.

7. A MODIFIED FRACTAL LATTICE

Since the model to which Lemma 4 applies is built up in blocks each containing a number of spins that is a power of 2, whereas in the fractal lattice F the number of spins in a block is a power of n that is not in general a power of 2, the lemma cannot be applied directly to F. However we can make it applicable by disconnecting some of the spins from F in a way that will now be described.

For reasons outlined in Section 1, we shall require $\alpha < 2D$. This condition is equivalent, by (3.1), to $\frac{1}{2}\alpha \log_2 m < \log_2 n$. It is therefore possible to find positive integers s, t such that

$$\tfrac{1}{2}\alpha \log_2 m < s/t \leqslant \log_2 n \tag{7.1}$$

and, in addition, such that

$$s \geqslant 2 \tag{7.2}$$

Multiplying (7.1) by t and exponentiating to base 2, we obtain

$$m^{\alpha t/2} < 2^s \leqslant n^t \tag{7.3}$$

When the fractal lattice F is built up by the process described in Section 2, the number of spins is multiplied at each step by n (which is the number of spins in the generating set A), and the magnification at each step is m. However, the same lattice can also be built up by using as generating set the set $B = A^{(t-1)}$ obtained by $t-1$ repetitions of the process, and as magnification factor the number m^t. The set B, which contains n^t sites, is defined according to (2.2) by

$$B = A^{(t-1)} = \{\mathbf{a} + m\mathbf{a}' + \cdots + m^{t-1}\mathbf{a}^{(t-1)}; \mathbf{a}, \mathbf{a}', ..., \mathbf{a}^{(t-1)} \in A\} \tag{7.4}$$

and we then have [in analogy with (2.2) and (2.3)]

$$F = \bigcup_{p=0}^{\infty} \{\mathbf{b} + m^t\mathbf{b}' + m^{2t}\mathbf{b}'' + \cdots + m^{pt}\mathbf{b}^{(p)}: \mathbf{b}, \mathbf{b}', ..., \mathbf{b}^{(p)} \in B\} \tag{7.5}$$

Let $\mathbf{u}, \mathbf{v}$ be any two sites in F; by (7.5) they can each be represented in the form analogous to (2.5)

$$\mathbf{u} = \sum_{p \geqslant 0} m^{tp}\mathbf{u}^{(p)}, \qquad \mathbf{v} = \sum_{p \geqslant 0} m^{tp}\mathbf{v}^{(p)}$$

where $\mathbf{u}^{(p)}$ and $\mathbf{v}^{(p)} \in B$ for all $p \geqslant 0$ and $\mathbf{u}^{(p)} = \mathbf{v}^{(p)} = \mathbf{0}$ for all sufficiently large p. To construct the Ising ferromagnet G to which Lemma 4 will be applied, let $C^{(0)}, C^{(1)}, C^{(2)}, ...$ be a family of subsets of B (not all of them will be different) chosen to satisfy the following conditions:

(i) $C^{(p)}$ comprises 2^s distinct sites.

(ii) $C^{(p)}$ includes $\mathbf{0}$, $\mathbf{u}^{(p)}$, and $\mathbf{v}^{(p)}$.

Condition (i) can be satisfied since, by (7.3), $2^s \leqslant n^t$ and B comprises n^t sites; condition (ii) can be satisfied since $\mathbf{0}$, $\mathbf{u}^{(p)}$, $\mathbf{v}^{(p)}$ are in B and, by (7.2), $s \geqslant 2$. Now we define, in analogy with (2.3) and (2.2),

$$G = \bigcup_{p=0}^{\infty} G^{(p)} \tag{7.6}$$

where

$$G^{(p)} = \{\mathbf{c}^{(0)} + m^t\mathbf{c}^{(1)} + \cdots + m^{pt}\mathbf{c}^{(p)}: \mathbf{c}^{(0)} \in C^{(0)}, \mathbf{c}^{(1)} \in C^{(1)}, ..., \mathbf{c}^{(p)} \in C^{(p)}\} \tag{7.7}$$

This construction ensures that **u** and **v** are both members of G. Moreover, since G is a subset of F, it follows by Griffiths' inequality that

$$\langle s(\mathbf{u})\, s(\mathbf{v})\rangle_F \geqslant \langle s(\mathbf{u})\, s(\mathbf{v})\rangle_G \tag{7.8}$$

where $\langle\cdots\rangle_F$ represents a thermal average on the fractal lattice F, as defined in Section 4, while $\langle\cdots\rangle_G$ represents the corresponding quantity for the lattice G, i.e., the case where the summation in (4.2) is restricted to those terms where $\{\mathbf{x}, \mathbf{y}\} \subset G^{(q)}$.

8. COMPARISON WITH A HIERARCHICAL MODEL

To apply Lemma 4 to the Ising ferromagnet on G, we need to put the sites of G into one-to-one correspondence with the nonnegative integers. First we label the sites in each of the finite sets $C^{(p)}$. To the integer 0 we associate the site $\mathbf{0}$ (calling it also $\mathbf{c}_0^{(p)}$) and to the integers 1, 2,..., 2^s-1 we associate the rest of the 2^s sites in $C^{(p)}$, in an arbitrary order, calling them $\mathbf{c}_1^{(p)}, ..., \mathbf{c}_{2^p-1}^{(p)}$. To find the site $\mathbf{g}_i$ in G corresponding to any given integer i, let the representation of i in the scale of 2^s be

$$i = \sum_{p\geqslant 0} i^{(p)} 2^{sp} \qquad \text{with} \quad i^{(p)} \in \{0, 1, ..., 2^s-1\} \tag{8.1}$$

Then we define $\mathbf{g}_i$ to be

$$\mathbf{g}_i = \sum_{p\geqslant 0} \mathbf{c}_{i^{(p)}}^{(p)} m^{tp} \tag{8.2}$$

which belongs to G because of (7.6) and (7.7). Provided (2.4) holds, it follows from Theorem 1 that every point in G has a unique representation in the form (8.2), and hence that the correspondence between nonnegative integers i and sites $\mathbf{g}_i$ in G is one-to-one.

We can now relate the "hierarchical distance" $d(i, j)$ between any two nonnegative integers i, j to the Euclidean distance between the corresponding sites $\mathbf{g}_i$, $\mathbf{g}_j$. Equation (8.2) implies

$$\begin{aligned} |\mathbf{g}_i - \mathbf{g}_j| &\leqslant \sum_{p=0}^{q} |\mathbf{c}_{i^{(p)}}^{(p)} - \mathbf{c}_{j^{(p)}}^{(p)}|\, m^{tp} \\ &\leqslant \sum_{p=0}^{q} R_{\max} m^{tp} \\ &= R_{\max}(m^{t(q+1)}-1)/(m^t-1) \end{aligned} \tag{8.3}$$

where q is the largest value of p such that $i^{(p)} \neq j^{(p)}$, and $R_{\max}$ is the diameter of the set B, of which every $C^{(p)}$ is a subset.

By the definition of q, we have $i^{(q)} \neq j^{(q)}$, so that when i and j are represented in the scale of 2^s, the $(q+1)$th-least significant digits in the two representations are different. So when i and j are represented in the scale of 2, at least one pair of corresponding digits between the $(sq+1)))$th- and the $s(q+1)$th-least significant digits (inclusive) are different, and it follows from the definition of $d(i, j)$ [Eq. (6.2)] that

$$d(i, j) \geqslant sq+1 \tag{8.4}$$

Combining (8.3) with (8.4), we obtain, since $m^t - 1 > 1$,

$$\begin{aligned} |\mathbf{g}_i - \mathbf{g}_j| &\leqslant R_{\max} m^t m^{t[d(i,j)-1]/s} \\ &= R_{\max} m^t 2^{[d(i,j)-1]\gamma/\alpha} \end{aligned} \tag{8.5}$$

where

$$\gamma = (t\alpha/s) \log_2 m \tag{8.6}$$

which satisfies, by (7.1),

$$\gamma < 2 \tag{8.7}$$

The interactions of an $r^{-\alpha}$ Ising ferromagnet on G therefore satisfy

$$|\mathbf{g}_i - \mathbf{g}_j|^{-\alpha} \geqslant (R_{\max} m^t)^{-\alpha} 2^\gamma 2^{-\gamma d(i,j)} \tag{8.8}$$

Equation (8.8) tells us that the interactions on the lattice G are stronger than those on a certain hierarchical model M of the type considered in Lemma 4. Hence, applying the Griffiths inequality[12] and then using Lemma 4 with $J_0 = (R_{\max} m^t)^{-\alpha} 2^\gamma$, we obtain [since $\gamma < 2$, by (8.7)]

$$\begin{aligned} \langle s(\mathbf{g}_i)\, s(\mathbf{g}_j) \rangle_G &\geqslant \langle s_i s_j \rangle_M \\ &\geqslant m^2 (T c_\gamma R_{\max}^\alpha m^{t\alpha}/2^\gamma) \end{aligned} \tag{8.9}$$

where c_γ is defined in (6.8).

Now, since $\mathbf{u}$ and $\mathbf{v}$ both belong to G, we can choose i and j so that $\mathbf{g}_i = \mathbf{u}$ and $\mathbf{g}_j = \mathbf{v}$; then (7.8) and (8.9) give

$$\langle s(\mathbf{u})\, s(\mathbf{v}) \rangle_F \geqslant m^2 (T c_\gamma R_{\max}^\alpha m^{t\alpha}/2^\gamma)$$

This lower bound is independent of the choice of $\mathbf{u}$ and $\mathbf{v}$ within F, and is positive provided

$$T < T_\gamma 2^\gamma / c^\gamma R_{\max}^\alpha m^{t\alpha}$$

where T_γ is the positive temperature whose existence [subject to the condition (8.7)] was proved by Dyson[11] [see Eq. (6.5) above]. So we have proved the following result.

Theorem 5. If $\alpha < 2D$, the Ising ferromagnet with $r^{-\alpha}$ interactions on a fractal lattice of dimension D satisfying the condition (2.4) has a spontaneous magnetization at sufficiently low temperatures.

Combining Theorems 3 and 5 gives our main result:

Theorem 6. If $D < \alpha < 2D$, the Ising ferromagnet with $r^{-\alpha}$ interactions on a fractal lattice of dimension D satisfying the condition (2.4) has a phase transition.

9. DISCUSSION

One of the questions raised by the result proved here is what happens when $\alpha > 2D$. The answer is that in this case the values of α and D alone are not sufficient to determine whether or not there is a phase transition. If D is an integer and the "fractal lattice" is simply the periodic lattice Z^D, then there is no transition in the case $\alpha > 2D$ for $D = 1$, but comparison with the nearest neighbor Ising model (using Griffiths' inequalities) shows that a phase transition does occur for $D \geqslant 2$. However, regardless of the value of D, it is easy to construct fractal lattices for which the condition $\alpha > 2D$ is sufficient to guarantee that there is no spontaneous magnetization for any positive T, and hence, presumably, no phase transition for any positive T. This can be achieved by strengthening the condition (2.4) to $m > 1 + r_{\max}/r_{\min}$, as is shown and proved in Theorem 7 in the Appendix. Unlike the periodic lattices, however, these fractal lattices must be embedded in a space of dimension larger than D.

One would also like to know more about the phase transition in the case $D < \alpha < 2D$; for example, what are the critical exponents? For periodic lattices, it was found by Fisher *et al.*,[16] using the renormalization group approach, that if $\alpha < \frac{3}{2}D$, the critical exponents have the "classical" values ($\eta = 2 - \sigma$, $\nu = 1/\sigma$, $\gamma = 1$, where $\sigma = \alpha - D$), but if $\alpha > \frac{3}{2}D$, they do not (and if $\alpha > D + 2$, they have the same values as for short-range foces). Similar results had been previously obtained analytically by Joyce[17] for the spherical model. It is possible that some of these results will also apply to the case of nonperiodic fractal lattices.

In this paper only the Ising model has been considered, but similar results can also be obtained for the plane rotator model: the proofs are analogous, using the inequalities of Ginibre[18] to compare the various models, the mean-field bound of Driessler *et al.*[19] to prove the absence of spontaneous magnetization at high temperatures, and some inequalities of Kunz and Pfister[10] to prove the presence of spontaneous magnetization at low temperatures. The conclusion is that there is a phase transition for the plane rotator model with $r^{-\alpha}$ interactions on a fractal lattice of dimension D if $D<\alpha<2D$. This generalizes the result of Kunz and Pfister, who proved this same thing for the plane square lattice.

APPENDIX

Theorem 7. An Ising ferromamgnet with $r^{-\alpha}$ interactions on a fractal lattice of dimension D for which $\alpha>2D$ and

$$m>1+r_{\max}/r_{\min} \tag{A.1}$$

has zero spontaneous mgnetization at all positive temperatures.

Proof. Let M denote the Ising ferromagnet to which the theorem refers, and for every positive integer q let $M^{(q)}$ denote the ferromagnet obtained from M by locking together all the spins in the subset $A^{(q)}$ defined in (2.2), and those in every translate of $A^{(q)}$ (a construction suggested by the work of Rogers and Thompson[20]). That is, if each site $\mathbf{x}$ in F is written in the form

$$\mathbf{x}=\mathbf{a}+m^{q+1}\mathbf{y}$$

with $\mathbf{a}\in A^{(q)}$ and $\mathbf{y}\in F$, then for each $\mathbf{y}$ all the spins with the same value of $\mathbf{y}$ are locked together. By the Griffiths inequality,[12] the spontaneous magnetization of the model $M^{(q)}$ is at least as great as that of M at the same temperature:

$$m(T)\leqslant m^{(q)}(T) \tag{A.2}$$

In the model $M^{(q)}$ each block of n^{q+1} spins behaves as one, and the interaction between the blocks $\mathbf{y}$ and $\mathbf{y}'$ is bounded above by

$$(n^{q+1})^2\, r(\mathbf{y},\mathbf{y}')^{-\alpha} \tag{A.3}$$

where

$$\begin{aligned} r(\mathbf{y},\mathbf{y}') &= \min_{\mathbf{a},\mathbf{b}\in A} |\mathbf{a}+m^{q+1}\mathbf{y}-\mathbf{b}-m^{q+1}\mathbf{y}'| \\ &\geqslant k_1 \min_{\mathbf{a},\mathbf{b}} \|\mathbf{a}+m^{q+1}\mathbf{y}-\mathbf{b}-m^{q+1}\mathbf{y}'\| \end{aligned} \tag{A.4}$$

with k_1 a positive geometrical constant relating the Euclidean norm to the norm $\|\cdot\|$ used in defining $r_{\max}$ and $r_{\min}$. From the definitions of $A^{(q)}$ and $r_{\max}$ we have

$$\|\mathbf{a}-\mathbf{b}\| \leqslant (1+m+\cdots+m^q)\, r_{\max} < \frac{m^{q+1}}{m-1} r_{\max}$$

and hence, from (A.4),

$$\begin{aligned} r(\mathbf{y}, \mathbf{y}') &> k_1 m^{q+1} \|\mathbf{y}-\mathbf{y}'\| \left[1 - \frac{r_{\max}}{(m-1)\|\mathbf{y}-\mathbf{y}'\|}\right] \\ &\geqslant k_1 k_2 m^{q+1} \|\mathbf{y}-\mathbf{y}'\| \end{aligned}$$

where

$$k_2 = 1 - r_{\max}/(m-1) r_{\min}$$

which is positive, by (A.1). The model $M^{(q)}$ is therefore equivalent to an Ising model on F in which the interaction between sites $\mathbf{y}$ and $\mathbf{y}'$ is [see (A.3)] less than $(n^2)^{q+1} r(\mathbf{y}, \mathbf{y}')^{-\alpha}$, which in turn is less than

$$J^*(\mathbf{y}, \mathbf{y}') = (n^2 m^{-\alpha})^{q+1} (k_1 k_2)^{-\alpha} \|\mathbf{y}-\mathbf{y}'\|^{-\alpha}$$

Therefore, by Griffiths' inequality, the spontaneous magnetization of $M^{(q)}$ is less than that for an Ising model M^* on F with the above interaction, which in turn is the same as the spontaneous magnetization for the original Ising model M at temperature

$$T^* = \frac{T}{(n^2 m^{-\alpha})^{q+1} (k_1 k_2)^{-\alpha}}$$

That is, $m^{(q)}(T) \leqslant m(T^*)$.

However, if $\alpha > 2D$, then $n^2 m^{-\alpha} < 1$ and hence T^* can be made (for any given T) as large as we please by making q large enough. In particular it can be made larger than the temperature above which Theorem 3 shows the spontaneous magnetization to be zero. So, whatever the value of T, we have $m^{(q)}(T) = 0$ for sufficiently large q, and it follows from (A.2) that $m(T) = 0$. ∎

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